Reliability Monitoring Using Log Gaussian Process Regression

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Traditional Regression and Bayesian Regression

• **Traditional Regression** assumes an underlying process (e.g., failure process described by the failure rate model $\lambda(t)$) which generates clean data. The goal is to describe the underlying model in the presence of noisy data.

• **Bayesian Regression** Specify the prior $P(H_\alpha)$ of a set of probabilistic models. The likelihood of $H_\alpha$ after observing data $D$ is $P(D \mid H_\alpha)$

The posterior probability of $H_\alpha$ is given by

$$P(H_\alpha \mid D) \propto P(H_\alpha)P(D \mid H_\alpha)$$

Prediction: $p(y \mid D) = \sum_\alpha P(y \mid H_\alpha)P(H_\alpha \mid D)$
GP Regression

• It is a nonlinear regression when you need to learn a function $f$ with uncertainties from data $D = \{X, y\}$

Ref: Eurandom 2010, Z. Ghahramani
GP Regression (Cont.)

• A Gaussian process defines a distribution over functions $p(f)$ which can be used for Bayesian regression

$$p(f|D) = p(f)p(D|f)/p(D)$$

• Gaussian processes (GPs) are parameterized by a mean function, $\mu(x)$, and a covariance function, or kernel, $K(x, x')$.

• The covariance matrix $K$ is between all the pair of points $x$ and $x'$ and specifies a distribution on functions

$$p(f(x), f(x')) = N(\mu, \Sigma)$$

$$\mu = \begin{bmatrix} \mu(x) \\ \mu(x') \end{bmatrix} \quad \Sigma = \begin{bmatrix} K(x, x) & K(x, x') \\ K(x', x) & K(x', x') \end{bmatrix}$$

and similarly for $p(f(x_1), \ldots, f(x_n))$ where now $\mu$ is an $n \times 1$ vector and $\Sigma$ is an $n \times n$ matrix
Imagine observing a data set $\mathcal{D} = \{(x_i, y_i)_{i=1}^{n}\} = (X, y)$.

Model:

$$y_i = f(x_i) + \epsilon_i$$

$$f \sim \text{GP}(\cdot | 0, K)$$

$$\epsilon_i \sim \text{N}(\cdot | 0, \sigma^2)$$

Prior on $f$ is a GP, likelihood is Gaussian, therefore posterior on $f$ is also a GP.

We can use this to make predictions

$$p(y_*|x_*, \mathcal{D}) = \int p(y_*|x_*, f, \mathcal{D}) p(f|\mathcal{D}) df$$

We can also compute the marginal likelihood (evidence) and use this to compare or tune covariance functions

$$p(y|X) = \int p(y|f, X) p(f) df$$
GP Regression (Cont.)

• Set of random variables, any finite collection of which follows joint Gaussian distribution

• Gaussian distribution specified by mean, \( \mu(x) \), and kernel function, \( k(x_1, x_2) \)

• May also be defined using standard linear model form

\[
y = f(x) + \varepsilon
\]
Kernel Function

- Kernel function = covariance function
- Kernel function determines correlation between data points
  - Can model trends in data such as periodicity
- Popular example is squared-exponential kernel function

$$k(x_1, x_2) = \sigma^2 \exp\left(\frac{||x_1 - x_2||^2}{2l^2}\right)$$

- $l$ is the characteristic length-scale of the process (showing, "how far apart" two points have to be for $X$ to change significantly)
- Used to develop overall covariance matrix $K$ for vector of data
Kernel Function Hyperparameters

- Kernel functions contain unknown hyperparameters
- Hyperparameters can be chosen by maximizing log-marginal likelihood given by

\[
\log p(y / X) = -\frac{1}{2} y^T K^{-1} y - \frac{1}{2} \log K - \frac{n}{2} \log 2\pi
\]

- Achieved using conjugate gradient optimization technique
  - Built-in option in available software packages
- Chi-square goodness-of-fit can also be applied to test data to assess model
Log Gaussian Processes

- Observed data Y are strictly positive
- Assume log(Y) is normally distributed

\[
\begin{bmatrix}
\log(Y) \\
\log(f^*)
\end{bmatrix} \sim N\left(\begin{bmatrix} a \\
b \end{bmatrix}, \begin{bmatrix} K(X, X) & K(X, X^*) \\
K(X^*, X) & K(X^*, X^*) \end{bmatrix}\right)
\]

- Can use conditional probability to determine prediction for \( f^* \)

\[
\log(f^*) / \log(Y) \sim N(\mu, \Sigma)
\]

\[
\mu = b + K(X^*, X)K(X, X)^{-1}(\log(Y) - b)
\]

\[
\Sigma = K(X^*, X^*) - K(X^*, X)^{-1}K(X, X^*)K(X, X^*)
\]

- Inverse transform can be used to proper domain
Application to Fleet of Vehicles

- Use Log GPR to model rate-of-occurrence of failures per month for fleet of vehicles
Covariance Function Choices

• Data appear to have periodic behavior along with noise

• Examine kernel function alternatives to describe correlation within data
  - Combinations of kernel functions are also kernel functions
  - Options include Squared Exponential (SE), Noise, Periodic, Polynomial, etc.

• Can use negative log-likelihood to discriminate between possible kernel functions
  - Smaller values indicate higher likelihood
  - Better description of data
## Kernel Function Likelihood

<table>
<thead>
<tr>
<th>Kernel Function Alternative</th>
<th>Description</th>
<th>Negative Log Likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SE, Periodic, Noise</td>
<td>7.84</td>
</tr>
<tr>
<td>2</td>
<td>SE, Noise</td>
<td>8.54</td>
</tr>
<tr>
<td>3</td>
<td>Noise</td>
<td>15.83</td>
</tr>
<tr>
<td>4</td>
<td>SE, Periodic</td>
<td>8.13</td>
</tr>
<tr>
<td>5</td>
<td>SE</td>
<td>8.54</td>
</tr>
<tr>
<td>6</td>
<td>SE, Polynomial, Noise</td>
<td>8.54</td>
</tr>
<tr>
<td>7</td>
<td>Polynomial</td>
<td>13.33</td>
</tr>
<tr>
<td>8</td>
<td>Rational quadratic, Noise</td>
<td>8.51</td>
</tr>
<tr>
<td>9</td>
<td>Rational quadratic</td>
<td>8.51</td>
</tr>
</tbody>
</table>

Sum of squared exponential, periodic, and noise kernels yields highest likelihood and best description of data.
Covariance Function and Hyperparameters

\[ k(x_1, x_2) = \theta_1^2 \exp\left[-\frac{(x_1 - x_2)^2}{2\theta_2^2}\right] + \theta_3^2 \exp\left[-\frac{2\sin^2\left(\frac{\pi (x_1 - x_2)}{\theta_4}\right)}{\theta_5^2}\right] + \sigma_n \delta(x_1, x_2) \]

\[ \delta(x_1, x_2) = \begin{cases} 1, & x_1 = x_2 \\ 0, & \text{o.w.} \end{cases} \]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_1 ) (SE 1)</td>
<td>4.38</td>
</tr>
<tr>
<td>( \theta_2 ) (SE 2)</td>
<td>0.31</td>
</tr>
<tr>
<td>( \theta_3 ) (Periodic 1)</td>
<td>0.13</td>
</tr>
<tr>
<td>( \theta_4 ) (Periodic 2)</td>
<td>1.03</td>
</tr>
<tr>
<td>( \theta_5 ) (Periodic 3)</td>
<td>0.26</td>
</tr>
<tr>
<td>( \sigma_n ) (Noise)</td>
<td>0.12</td>
</tr>
</tbody>
</table>
Predicted Result

Model provides probabilistic prediction of failures in future months.

“+” indicates test data used in goodness-of-fit test.

**Dashed lines indicate 95% probability interval**
Anomaly Detection Example

**Dashed lines indicate 95% probability interval**
Conclusions and Possible Extensions

• LOG GP Regression is powerful technique for modeling complex nonlinear behavior
  • Provides probabilistic indication of reliability problems vs. typical trends within data
• Can be extended to model more complex nonlinear relationships
  • Only require appropriate kernel function
• Can also handle multidimensional inputs
  • Systems in different locations, different ages, etc.